UCL Math 1101 (Analysis 1) — Fall 2011 — C. Wendl

HOMEWORK 1
Due: 10 October, 2011

Problems marked with $(*)$ will be graded. Students are advised to work through ALL problems in the assignments, so as to keep pace with the material.

1. Find all real values of x such that
   (a) $\dfrac{x+1}{x^2+3} < \dfrac{2}{x}$
   (b) $\left|\dfrac{1}{x+1} - 1\right| < 2$
   (c) $(*)$ For these next two, explain your answers graphically by plotting carefully the function $g(x) = |x+3| + |5-2x|$:
      i. $|x+3| + |5-2x| \le 4$
      ii. $|x+3| + |5-2x| \le 6$
   (d) $(*)$ $x - \big|x - |x|\big| > 2$.

2. $(*)$ Please note that in the following problem, it may be possible to answer part (b) even if you are unable to figure out part (a).
   (a) Let q be a natural number. Show that if q is not divisible by 3, then neither is q^2. *Hint: If q is not divisible by 3, what are the possible remainders when we divide q by 3? There are two cases to consider.*
   (b) Assuming the statement that was to be proved in part (a), deduce that $\sqrt{3}$ is irrational.

3. $(*)$ Suppose x is an irrational number and r is rational. Show that:
   (a) $r + x$ is irrational.
   (b) if, moreover, $r \neq 0$, then rx is also irrational.

   Hint: you can prove the results by contradiction, e.g. if you were to assume that $r + x$ is rational, what would this imply for x?

4. Prove that for all real numbers $x \neq 0$,
$$\left|\frac{1}{x}\right| = \frac{1}{|x|}.$$

5. (hard, but worthwhile) The following exercise explains a nice trick for finding rational approximations to irrational square roots—it was known hundreds of years before the invention of the calculator. In this example, our goal will be to approximate $\sqrt{3}$. We start with the pair of integers $(x_1, y_1) = (2, 1)$. Next, we can recursively define for $n = 2, 3, 4, \ldots$ pairs of integers
$$(x_{n+1}, y_{n+1}) = (2x_n + 3y_n, x_n + 2y_n) \quad \text{i.e.} \quad x_{n+1} = 2x_n + 3y_n, \quad y_{n+1} = x_n + 2y_n.$$
   (a) Show that $(x_2, y_2) = (7, 4)$, $(x_3, y_3) = (26, 15)$, $(x_4, y_4) = (97, 56)$, $(x_5, y_5) = (362, 209)$.
   (b) Prove that the numbers y_n always increase with n. (In fact, one can show that they become *arbitrarily* large as n increases, but don't worry about this detail unless you really feel like it.)
   (c) Prove that if the pair of integers (x_n, y_n) satisfies
$$x_n^2 - 3y_n^2 = 1,$$
   then
$$x_{n+1}^2 - 3y_{n+1}^2 = 1.$$
   Observe that this equation is indeed satisfied for $n = 1$, so by induction, it is satisfied for all n.

(d) Conclude from the previous step that $\sqrt{3} < \frac{x_n}{y_n}$ and $\left(\frac{x_n}{y_n}\right)^2 = 3 + \frac{1}{y_n^2}$ for all n.

(e) Show that $\frac{x_{n+1}}{y_{n+1}} < \frac{x_n}{y_n}$ for all n.

(f) Combining the above steps, conclude that the fractions x_n/y_n yield a decreasing sequence of rational numbers that come closer and closer to $\sqrt{3}$ as n gets larger:

$$\frac{2}{1} > \frac{7}{4} > \frac{26}{15} > \frac{97}{56} > \frac{362}{209} > \cdots > \sqrt{3}.$$

(Actually, using the fact that y_n becomes arbitrarily large, one can in fact show that x_n/y_n becomes *arbitrarily* close to $\sqrt{3}$ as n increases. The equations $x^2 - 3y^2 = 1$ and $x^2 - 2y^2 = \pm 1$ are two of the so-called *Pell's equations*.)

UCL Math 1101 (Analysis 1) — Fall 2011 — C. Wendl

HOMEWORK 2
Due: 17 October, 2011

Problems marked with $(*)$ will be graded. Students are advised to work through ALL problems in the assignments, so as to keep pace with the material.

1. $(*)$ In the following, you may find it useful to argue by induction.
   (a) Prove that for all $x \geq -1$ and $n \in \mathbb{N}$, $(1+x)^n \geq 1+nx$. (This is called Bernoulli's inequality.) Deduce from this that $2^n > n$ for $n \in \mathbb{N}$.
   (b) Prove that for all $x \geq -1$ with $x \neq 0$ and all integers $n \geq 2$, $(1+x)^n > 1+nx$. Is this also true for $x = 0$ or for $n = 1$?

2. (a) (*) Use the Archimedean property to prove that for any real number $\epsilon > 0$, there exists a natural number $n \in \mathbb{N}$ such that
   $$\frac{1}{n} < \epsilon.$$
   You may assume the fact (proved in lecture) that for any two positive real numbers a and b, $a > b$ if and only if $1/a < 1/b$.
   (b) Consider the set $S = \{x \in \mathbb{R} : x^2 < 2\}$. As we discussed in lecture, the continuum property of the real numbers implies that since S is bounded above, it has a *least upper bound*, which in this case is $\sqrt{2}$. Show that if H is any upper bound for S that is a *rational* number, then there exists another rational number $h \in \mathbb{Q}$ which is also an upper bound for S but satisfies $h < H$. (This implies that S has no least upper bound in $\mathbb{Q}$, thus unlike $\mathbb{R}$, $\mathbb{Q}$ does *not* have the contimuum property!)
   Hint: you might find the result of part (a) useful for part (b), and you may also want to use the fact that $\sqrt{2}$ is irrational.

3. Each of the following denotes a subset of $\mathbb{R}$. Find the sup, inf, max and min of each, or state when any of these do not exist.[1]
   (a) $[1, 4)$
   (b) (*) $[-2, \infty)$
   (c) (*) $\{0, 1, 3, -6\}$
   (d) $[3, 5]$
   (e) $\{x : x^2 - 5x + 6 < 0\}$
   (f) (*) $\{x^2 - 5x + 6 : x \in \mathbb{R}\}$. (Careful: make sure you properly understand what this set is before you answer the question!)
   (g) $\{x : x^2 + 1 = 0\}$
   (h) $\{2^{-k} + 3^{-m} : k, m \in \mathbb{N}\}$. (Provide careful explanations for the infimum; you may find Problem 2(a) helpful for this.)

4. $(*)$ Suppose A and B are two nonempty sets of real numbers. We can define their *sum* as the set
   $$A + B = \{a + b : a \in A \text{ and } b \in B\},$$
   i.e. we include in $A + B$ the sums of elements of A with elements of B in all possible ways. If $A + B$ is assumed to have upper bound $M \in \mathbb{R}$, show that:

[1] A quick remark about symbols: the notation used for the sets in Problems 3 and 4 matches the style of notation in Binmore, which is slightly different from what we've used in the lectures. For example, where Binmore writes $\{x : x^2 < 2\}$, we would have written $\{x \mid x^2 < 2\}$ in lectures, and many other books would write $\{x \,;\, x^2 < 2\}$. These three variations all have the same meaning, namely "all numbers (in the present context we mean *real* numbers) x such that $x^2 < 2$." Since all three variations are in common usage, it's important to become accustomed to all of them.

(a) A and B are each bounded above.

(b) $\sup A + \sup B \leq M$.

5. Prove by induction that $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

UCL Math 1101 (Analysis 1) — Fall 2011 — C. Wendl

HOMEWORK 3 SOLUTIONS
Due: 24 October, 2011

Problems marked with $(*)$ will be graded. Students are advised to work through ALL problems in the assignments, so as to keep pace with the material.

1. Use the Cauchy-Schwarz inequality to show each of the following:

   (a) (*) For any real numbers $x_1, x_2, \ldots, x_n$,

$$\left(\sum_{i=1}^n x_i\right)^2 \leq n \sum_{i=1}^n x_i^2.$$

   We recall the Cauchy-Schwarz inequality:

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2.$$

   We apply it here with $a_i = x_i$, and $b_i = 1$, $i = 1, \ldots, n$. Then we find,

$$\left(\sum_{i=1}^n x_i \cdot 1\right)^2 \leq \sum_{i=1}^n x_i^2 \sum_{i=1}^n 1^2 = \sum_{i=1}^n x_i^2 \cdot n,$$

   since in the last sum we have n summands all equal to 1.

   (b) (*) For any positive real numbers a, b, c,

$$\left(\frac{1}{2}a + \frac{1}{3}b + \frac{1}{6}c\right)^2 \leq \frac{1}{2}a^2 + \frac{1}{3}b^2 + \frac{1}{6}c^2.$$

   We apply Cauchy-Schwarz with $n = 3$, and

$$a_1 = \frac{1}{\sqrt{2}}a, \quad a_2 = \frac{1}{\sqrt{3}}b, \quad a_3 = \frac{1}{\sqrt{6}}c, \quad b_1 = \frac{1}{\sqrt{2}}, \quad b_2 = \frac{1}{\sqrt{3}}, \quad b_3 = \frac{1}{\sqrt{6}}.$$

   The Cauchy-Schwarz inequality then gives

$$\left(\frac{1}{2}a + \frac{1}{3}b + \frac{1}{6}c\right)^2 \leq \left(\frac{1}{2}a^2 + \frac{1}{3}b^2 + \frac{1}{6}c^2\right)\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) = \left(\frac{1}{2}a^2 + \frac{1}{3}b^2 + \frac{1}{6}c^2\right),$$

   since $1/2 + 1/3 + 1/6 = 1$.

   (c) For any positive real numbers $x_1, x_2, \ldots, x_n$,

$$n^2 \leq (x_1 + x_2 + \cdots x_n) \cdot \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right).$$

   We apply the Cauchy-Schwarz inequality with

$$a_1 = \sqrt{x_1}, \quad a_2 = \sqrt{x_2}, \quad \ldots \quad a_n = \sqrt{x_n},$$
$$b_1 = \frac{1}{\sqrt{x_1}}, \quad b_2 = \frac{1}{\sqrt{x_2}}, \quad \ldots \quad b_n = \frac{1}{\sqrt{x_n}}.$$

   We have

$$a_1 b_1 = 1, \quad a_2 b_2 = 1, \quad \ldots \quad a_n b_n = 1.$$

   so that

$$\begin{aligned} n^2 = (1 + 1 + \cdots + 1)^2 = (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 &\leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \\ &= (x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right). \end{aligned}$$

2. Using the definition of convergence (not the sandwich or the combination theorem), prove:

(a) (*) $\lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}$

We need to show that given any $\epsilon > 0$, we can find a number N, such that

$$n > N \Longrightarrow \left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \epsilon.$$

We first rewrite the expression inside the absolute values as

$$\frac{3n+1}{2n+5} - \frac{3}{2} = \frac{2(3n+1)-3(2n+5)}{2(2n+5)} = \frac{6n+2-6n-15}{2(2n+5)} = \frac{-13}{2(2n+5)}.$$

Next we try to determine which natural numbers n will satisfy

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \frac{13}{2(2n+5)} < \epsilon \Leftrightarrow \frac{2(2n+5)}{13} > \frac{1}{\epsilon} \Leftrightarrow 2n+5 > \frac{13}{2\epsilon} \Leftrightarrow 2n > \frac{13}{2\epsilon} - 5$$

$$\Leftrightarrow n > \frac{1}{2}\left(\frac{13}{2\epsilon} - 5\right).$$

Thus it suffices to take

$$N = \frac{1}{2}\left(\frac{13}{2\epsilon} - 5\right).$$

(b) (*) $\lim_{n\to\infty} \frac{\cos n}{n} = 0$

Given $\epsilon > 0$, we need to find a number N such that

$$n > N \Longrightarrow \left|\frac{\cos n}{n}\right| < \epsilon.$$

It is difficult to solve the equation $\cos(N)/N = \epsilon$ for N. Moreover, the sequence $\cos(n)/n$ is not decreasing, so we cannot guarantee $\cos(n)/n < \cos(N)/N$ for $n > N$. Even worse, the terms are not all positive:

$$\frac{\cos 1}{1} = 0.54030, \quad \frac{\cos 2}{2} = -0.2080, \quad \frac{\cos 3}{3} = -.32999, \quad \frac{\cos 4}{4} = -0.1634, \quad \frac{\cos 5}{5} = 0.0567.$$

However, we notice that the numerator is $\cos(n)$ and the function $\cos(x)$ is always bounded between -1 and 1, i.e. $|\cos(x)| \leq 1$. This gives us

$$\left|\frac{\cos(n)}{n}\right| = \frac{|\cos(n)|}{n} \leq \frac{1}{n}.$$

Now to make the left hand side less than ϵ, we observe (cf. Homework 2, Problem 2(a)) that $1/n < \epsilon$ if and only if $n > 1/\epsilon$. Thus it suffices to take $N = 1/\epsilon$, as then

$$n > N \Longrightarrow \frac{1}{n} < \frac{1}{N} = \epsilon \Longrightarrow \left|\frac{\cos(n)}{n}\right| \leq \frac{1}{n} < \epsilon.$$

(c) $\lim_{n\to\infty} \frac{5\cdot 2^n - 4}{2^n - 1} = 5$

Hint: you may want to use the fact that $2^n > n$ for all $n \in \mathbb{N}$, proved in Homework 2, Problem 1(a).

We have

$$\frac{5\cdot 2^n - 4}{2^n - 1} - 5 = \frac{5\cdot 2^n - 4}{2^n - 1} - \frac{5(2^n-1)}{2^n - 1} = \frac{5\cdot 2^n - 4 - 5\cdot 2^n + 5}{2^n - 1} = \frac{1}{2^n - 1}.$$

Abbreviate $x_n := \frac{5\cdot 2^n - 4}{2^n - 1}$; then for any given $\epsilon > 0$, we need to find $N > 0$ such that $n > N \Longrightarrow |x_n - 5| < \epsilon$. By the above calculation, $|x_n - 5| < \epsilon$ is equivalent to

$$\left|\frac{1}{2^n - 1}\right| < \epsilon \Leftrightarrow \frac{1}{2^n - 1} < \epsilon \Leftrightarrow 2^n - 1 > \frac{1}{\epsilon}$$

$$\Leftrightarrow 2^n > \frac{1}{\epsilon} + 1. \tag{1}$$

Thus it suffices to have $2^n > \dfrac{1}{\epsilon} + 1$. Since $2^n > n$ by Problem 1(a) in Homework 2, it will suffice in fact to have

$$n > \frac{1}{\epsilon} + 1, \tag{2}$$

thus we take $N = 1/\epsilon + 1$. Then $n > N$ implies (2) and this implies (1), which is equivalent to $|x_n - 5| < \epsilon$.

3. Find the following limits. For these you may use any facts or techniques that have been discussed in the lectures, or in a previous homework, or in Binmore.

(a) $\displaystyle\lim_{n\to\infty} \frac{4n^5 + 5n^3 + 6n}{2n^5 + 1}$

We factor n^5 from numerator and denominator to get

$$\lim_{n\to\infty} \frac{n^5(1 + 2/n^2 + 6/n^4)}{n^5(2 + 1/n^5)} = \lim \frac{1 + 2/n^2 + 6/n^4}{2 + 1/n^5} = \frac{\lim 1 + \lim(2/n^2) + \lim(6/n^4)}{\lim 2 + \lim(1/n^5)},$$

using the combination theorem (Binmore Proposition 4.8, (i) and (iii)). Since $\lim(1/n^r) = 0$ for any positive rational number r (as shown in lecture; see also Exercise 4.6(2) in Binmore), another application of the combination theorem gives $\lim(c/n^r) = 0$ for any positive $r \in \mathbb{Q}$ and any constant $c \in \mathbb{R}$. Thus every limit of this form in the above expression vanishes, and we are left with

$$\lim_{n\to\infty} \frac{n^5 + 2n^3 + 6n}{2n^5 + 1} = \frac{\lim 1 + \lim(2/n^2) + \lim(6/n^4)}{\lim 2 + \lim(1/n^5)} = \frac{1 + 0 + 0}{2 + 0 + 0} = \frac{1}{2}.$$

Here we have also used the obvious fact that a constant sequence $x_n = c$ converges to c.

(b) (*) $\displaystyle\lim_{n\to\infty} \frac{3^n + 2^n}{3^n + (-2)^n}$

We factor 3^n from both numerator and denominator to get:

$$\lim_{n\to\infty} \frac{3^n + 2^n}{3^n + (-2)^n} = \lim \frac{3^n(1 + (2/3)^n)}{3^n(1 + (-2/3)^n)} = \lim \frac{1 + (2/3)^n}{1 + (-2/3)^n}$$

$$= \frac{\lim 1 + \lim(2/3)^n}{\lim 1 + \lim(-2/3)^n} = \frac{1 + 0}{1 - 0} = 1,$$

where we have used the combination theorem and the fact that $\lim(x^n) = 0$ for any $x \in \mathbb{R}$ with $|x| < 1$ (shown in lecture; see also Binmore Example 4.12).

(c) $\displaystyle\lim_{n\to\infty} \left(\frac{n^2}{n^3} + \frac{(n+1)^2}{n^3} + \cdots + \frac{(2n)^2}{n^3}\right)$.

We notice that the numerators are the squares of the natural numbers from n to $2n$. To sum them, we make use of Homework 2 Problem 5 and write

$$n^2 + (n+1)^2 + \cdots + (2n)^2 = (1^2 + 2^2 + \cdots + n^2 + \cdots + (2n)^2) - (1^2 + 2^2 + \cdots (n-1)^2)$$

$$= \frac{2n(2n+1)(4n+1)}{6} - \frac{(n-1)n(2n-2+1)}{6}.$$

This gives

$$\lim_{n\to\infty}\left(\frac{n^2}{n^3}+\frac{(n+1)^2}{n^3}+\cdots+\frac{(2n)^2}{n^3}\right)=\lim\left(\frac{2n(2n+1)(4n+1)}{6n^3}-\frac{(n-1)n(2n-1)}{6n^3}\right)$$

$$=\lim\frac{2n^3(2+1/n)(4+1/n)}{6n^3}-\lim\frac{(1-1/n)n^3\cdot(2-1/n)}{6n^3}$$

$$=\lim\frac{1}{3}(2+1/n)(4+1/n)-\lim\frac{1}{6}(1-1/n)(2-1/n)=\frac{1}{3}\cdot 2\cdot 4-\frac{1}{6}\cdot 1\cdot 2=\frac{8}{3}-\frac{1}{3}=\frac{7}{3}.$$

4. Prove the following version of the principle of induction: Suppose that for each $n \in \mathbb{N}$, $P(n)$ is a statement about the natural number n, and it is known that

   (i) $P(1)$ and $P(2)$ are both true,

   (ii) for each $n \in \mathbb{N}$, if $P(n)$ and $P(n+1)$ are both true, then $P(n+2)$ is true.

   Then $P(n)$ is true for every $n \in \mathbb{N}$.

   For a hint as to how such a proof might go, see Theorem 3.8 in Binmore.

   Let $S = \{n \in \mathbb{N} \mid P(n) \text{ is false }\}$. We want to prove that S is empty. We argue by contradiction: suppose S is not empty. By (i), $1 \notin S$ and $2 \notin S$. By the well ordering principle, every nonempty subset of $\mathbb{N}$ has a minimum, thus in particular S has a minimum, call it m. Then $m \in S$, so $m \geq 3$ since 1 and 2 are not in S. It follows that $m-1$ and $m-2$ are both natural numbers, and since m is the minimum of S, neither $m-1$ nor $m-2$ is in S. This means $P(m-1)$ and $P(m-2)$ are both true, but then by (ii), $P(m)$ has to be true, so $m \notin S$. This is a contradiction, hence our assumption was false: S must be empty after all.

5. The Fibonacci sequence is defined by

$$x_1 = 1, \quad x_2 = 1, \quad x_{n+2} = x_{n+1} + x_n, \quad n \geq 2.$$

   (a) Prove, using the previous problem, that

$$x_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}, \quad n \in \mathbb{N}.$$

   The statement $P(n)$ is the equation for x_n above. We check $P(1)$ and $P(2)$:

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 = x_1,$$

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} = \frac{\left(\frac{1+2\sqrt{5}+5}{4}\right) - \left(\frac{1-2\sqrt{5}+5}{4}\right)}{\sqrt{5}} = \frac{4\sqrt{5}/4}{\sqrt{5}} = 1 = x_2.$$

   Now we assume the validity of the formula for n and $n+1$, i.e.

$$x_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}, \quad x_{n+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}.$$

   We need to prove that

$$x_{n+2} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}}{\sqrt{5}}.$$

By the recursion formula, we have

$$x_{n+2} = x_n + x_{n+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n \left(1 + \frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n \left(1 + \frac{1-\sqrt{5}}{2}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n \left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n \left(\frac{3-\sqrt{5}}{2}\right)\right].$$

The tricky part now is to notice that to get the right formula, one needs

$$\frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2, \quad \frac{3-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2.$$

As it happens, this is true:

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2},$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2}.$$

This completes the inductive proof.

(b) (*) The number $\sqrt{5}$ is irrational. (You probably know enough to prove this by now—in which case you are encouraged to do so in your spare time—but for now you may simply assume it.) Is the number x_n given by the formula above *rational* for all n? Explain your answer.

It may not *look* rational at first, but it is; in fact, x_n is always a *natural* number. This is easy to prove from the recursion formula: it's clearly true for $n = 1$ and $n = 2$, and if we assume $x_n, x_{n+1} \in \mathbb{N}$, then obviously $x_{n+2} \in \mathbb{N}$ as well, since the sum of two natural numbers is natural. The modified inductive principle of Problem 4 then implies that $x_n \in \mathbb{N}$ for all n.

Now you may wonder: how can x_n be a natural number when the above formula contains no less than three appearances of the irrational number $\sqrt{5}$? You may have already developed some intuition for this if you did the necessary computation in part (a) to show that $x_1 = x_2 = 1$: in both of those cases, all instances of $\sqrt{5}$ cancel out in the end. That's what happens in general, and if we didn't already know it must be so due to the above inductive argument, we could also prove it using the *binomial theorem*. Try the computation for $n = 3$ and $n = 4$, and see if you can recognize a pattern.

UCL Math 1101 (Analysis 1) — Fall 2011 — C. Wendl

HOMEWORK 4
Due: 31 October, 2011

Problems marked with $(*)$ will be graded. Students are advised to work through ALL problems in the assignments, so as to keep pace with the material.

A general note: unless otherwise specified, in each problem you may use any facts or techniques that have been discussed in the lectures, or in a previous homework, or in Binmore.

1. (*) Find the limit of the sequence
$$x_n = \sqrt[n]{3^n + 5^n + 7^n}$$
and explain your answer.

2. Consider the sequence defined recursively by $x_1 = 1$ and $x_{n+1} = \sqrt{4x_n + 5}$.
   (a) (*) Is it a monotone sequence, and if so, is it increasing or decreasing? Prove your answer!
   (b) (*) Determine whether the sequence converges, and if so, find its limit.
   (c) Consider the Fibonacci sequence, which is defined recursively by $x_1 = x_2 = 1$ and $x_{n+2} = x_n + x_{n+1}$ for all $n \in \mathbb{N}$. One might try to compute a limit of this sequence as follows: if $\lim x_n = l$, then the combination theorem implies
$$l = \lim x_{n+2} = \lim (x_n + x_{n+1}) = \lim x_n + \lim x_{n+1} = l + l = 2l,$$
   thus $l = 2l$, which can only be true if $l = 0$. The careless mathematician would then conclude $x_n \to 0$. Explain why this is nonsense.

3. Prove that if $\langle x_n \rangle$ is a sequence with $x_n \to +\infty$ or $x_n \to -\infty$, then
$$\frac{1}{x_n} \to 0.$$
The converse is not true: find an example of a sequence $\langle x_n \rangle$ such that $\frac{1}{x_n} \to 0$ but neither $x_n \to +\infty$ nor $x_n \to -\infty$ is true.

4. Suppose k is a constant with $0 < k < 1$, and $\langle x_n \rangle$ is a sequence satisfying
$$|x_{n+1}| < k|x_n|$$
for every $n \in \mathbb{N}$. Prove that $x_n \to 0$ as $n \to \infty$. Explain why the same conclusion holds if it is only known that for some fixed $N \in \mathbb{N}$, $|x_{n+1}| < k|x_n|$ for all $n > N$.

5. Let $S \subset \mathbb{R}$ be any set of real numbers.
   (a) (*) Show that if S is bounded below, then there exists a sequence $\langle x_n \rangle$ such that $x_n \in S$ for all n and $x_n \to \inf S$.
   Hint: for any $n \in \mathbb{N}$, there is a number $x \in S$ such that $x < \inf S + \frac{1}{n}$. (Why?)
   (b) (*) Show that if S is unbounded below, then there exists a sequence $\langle x_n \rangle$ such that $x_n \in S$ for all n and x_n diverges to $-\infty$, i.e. $x_n \to -\infty$.
   (c) For each of the two statements above, there is a corresponding statement concerning upper bounds and $\sup S$ or convergence to $+\infty$. Write down both statements. (You needn't write down the proofs, which are completely analogous.)

6. Recall that for two positive real numbers $a, b > 0$, we have defined two types of *averages*, namely the arithmetic mean

$$A = \frac{a+b}{2}$$

and the geometric mean

$$G = \sqrt{ab}.$$

These are in general not equal, and in fact, an easy case of the inequality of the arithmetic and geometric mean (proved in lecture on 13/10/2011; see also Binmore 3.10) states that $G \leq A$. The following idea for finding a "compromise" between the arithmetic and geometric mean was discovered in the late 18th century and is usually credited to Gauss, though it was also discovered independently by Lagrange.

We shall assume without loss of generality that $a \leq b$ and, for reasons that will become apparent shortly, write $G_1 := a$ and $A_1 := b$. Now as a first step, denote by A_2 and G_2 the two means of a and b defined above. Since in general $G_2 \leq A_2$ but the two need not be equal, we proceed by averaging the two averages, again in both ways: define

$$A_3 = \frac{A_2 + G_2}{2}, \qquad G_3 = \sqrt{A_2 G_2}.$$

Now $G_3 \leq A_3$, and again the two need not be equal, so we average them again. Repeating these steps indefinitely, we recursively define a pair of sequences $\langle A_n \rangle$ and $\langle G_n \rangle$ such that

$$G_1 = a, \quad A_1 = b, \qquad G_{n+1} = \sqrt{G_n A_n}, \quad A_{n+1} = \frac{G_n + A_n}{2}.$$

The inequality of the arithmetic and geometric mean guarantees that $G_n \leq A_n$ for every $n \in \mathbb{N}$.

(a) Show that both sequences are monotone: in particular, G_n is increasing and A_n is decreasing. Conclude that both sequences converge.

(b) Prove that $\lim G_n = \lim A_n$. This common limit is called the *arithmetic-geometric mean* of a and b, and sometimes written

$$M(a, b) := \lim G_n = \lim A_n.$$

UCL Math 1101 (Analysis 1) — Fall 2011 — C. Wendl

HOMEWORK 5
Due: 14 November, 2011

Problems marked with $(*)$ will be graded. Students are advised to work through ALL problems in the assignments, so as to keep pace with the material.

1. (*) Show that the following sequences diverge. *Hint: think about subsequences.*

   (a) $a_n = 1 + (-1)^n + \dfrac{1}{n}$

   (b) $b_n = \sin\left(\dfrac{n\pi}{4}\right)$

2. Given a sequence $\langle x_n \rangle$, one can define a new sequence $\langle a_n \rangle$ by

$$a_n = \frac{x_1 + x_2 + \cdots + x_n}{n},$$

   i.e. for each n, a_n is the average of the first n terms of the original sequence.

   (a) (*) Prove that if $x_n \to x$ then $a_n \to x$ as well.

   (b) Though it may seem surprising at first, it is possible that the sequence of averages $\langle a_n \rangle$ defined above may converge even if the original sequence $\langle x_n \rangle$ does not. Show that this is so if we take $\langle x_n \rangle$ to be the sequence $0, 1, 0, 1, 0, 1, \ldots$. What is $\lim a_n$ in this case?

   (c) Find an example of a sequence $\langle x_n \rangle$ that diverges such that $\langle a_n \rangle$ also diverges.

3. Consider the sequence $\langle a_n \rangle$ defined recursively by $a_1 = 0$ and

$$a_{2n} = \frac{a_{2n-1}}{2}, \quad a_{2n+1} = \frac{1}{2} + a_{2n}, \quad n \in \mathbb{N}.$$

   From these, one can derive recurrence relations for the subsequences of even/odd terms: we have

$$a_{2n+2} = \frac{a_{2n+1}}{2} = \frac{\left(\frac{1}{2} + a_{2n}\right)}{2} = \frac{1}{4} + \frac{a_{2n}}{2}, \quad a_{2n+1} = \frac{1}{2} + a_{2n} = \frac{1}{2} + \frac{a_{2n-1}}{2}.$$

   (a) Show that the subsequence $\langle a_{2n} \rangle$ is increasing and bounded above by $1/2$. Compute its limit.

   (b) Show that the subsequence $\langle a_{2n-1} \rangle$ is increasing and bounded above by 1. Compute its limit.

   (c) Is the sequence $\langle a_n \rangle$ convergent?

4. Consider the sequence defined by

$$x_1 = 1, \quad x_{n+1} = \frac{1}{1 + x_n}, \quad n = 1, 2, \ldots.$$

   For instance:

$$x_2 = \frac{1}{1 + \frac{1}{1}}, \qquad x_3 = \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}, \quad x_4 = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}.$$

   Such expressions are called *continued fractions*. Let $l = (\sqrt{5} - 1)/2$, which is the unique positive root of the quadratic equation $x^2 + x - 1 = 0$.

   (a) Show inductively that $x_{2n} < l$ and $x_{2n-1} > l$.

   (b) Show inductively that the subsequence $\langle x_{2n} \rangle$ is increasing, while the subsequence $\langle x_{2n-1} \rangle$ is decreasing.

(c) Show that $\lim x_n = l$.

(d) Define two sequences of non-negative integers:

$$p_0 = 0, \quad p_1 = 1, \quad p_{n+2} = p_{n+1} + p_n, \quad q_0 = 1, \quad q_1 = 1, \quad q_{n+2} = q_{n+1} + q_n.$$

Prove that $q_n = p_{n+1}$ and $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$.

(e) Prove that $x_n = p_n/q_n$ for every n, and that there is no integer (apart from ± 1) dividing both the numerator and denominator of p_n/q_n, i.e. p_n and q_n have no common factors.

5. (*) Suppose $\langle x_n \rangle$ is a sequence satisfying

$$|x_{n+1} - x_n| < \frac{1}{2^n}$$

for each n. Show that x_n converges.

Hint: do not try to compute $\lim x_n$, as you don't have enough information to do it. What general situations have you learned about in which a sequence is guaranteed to converge?

6. A set S of real numbers is called *countable* if its elements can be arranged into a sequence, i.e. there exists a sequence $\langle x_n \rangle$ such that for every element $a \in S$, there is a unique $n \in \mathbb{N}$ with $x_n = a$. Obviously the natural numbers $\mathbb{N}$ are countable, and the integers $\mathbb{Z}$ are as well since one can arrange them into the sequence $0, 1, -1, 2, -2, 3, -3, \ldots$. It's not hard to show that even the rational numbers $\mathbb{Q}$ are countable; we won't go into the details here, but if you're curious you can find a standard and fairly easy proof for instance at:

`http://www.homeschoolmath.net/teaching/rational-numbers-countable.php`

In the following problem, we will show that unlike all the sets mentioned above, the *real* numbers $\mathbb{R}$ are not countable. In particular, we will show that no sequence can include every real number in its range. (In this sense, one can intuitively think of $\mathbb{R}$ as being a "larger" set than $\mathbb{Q}$, $\mathbb{Z}$ or $\mathbb{N}$, even though all of them are infinite!)

(a) (*) Show that if $a < b$ and $\langle x_n \rangle$ is a convergent sequence such that $x_n \in [a, b]$ for all n, then the limit of the sequence is also in $[a, b]$. You may use the fact that for any constant $c \in \mathbb{R}$, if $x_n \geq c$ (or $x_n \leq c$) for all n then $\lim x_n \geq c$ (or $\lim x_n \leq c$). Give an example to show that if x_n is only assumed to belong to the *open* interval (a, b) for every n, then $\lim x_n$ need not belong to (a, b).

(b) (*) Suppose $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences with the property that for every $n \in \mathbb{N}$, $a_n < b_n$ and

$$[a_{n+1}, b_{n+1}] \subset [a_n, b_n].$$

Show then that the set

$$\{x \in \mathbb{R} \mid x \in [a_n, b_n] \text{ for every } n \in \mathbb{N}\}$$

is not empty. Give an example to show that there need not exist any number that is contained in all the *open* intervals (a_n, b_n).

Hint: since each of the intervals $[a_n, b_n]$ is nonempty, there is a sequence $\langle x_n \rangle$ with $x_n \in [a_n, b_n]$ for all n. Then for any $k \in \mathbb{N}$, the subsequence $x_k, x_{k+1}, x_{k+2}, \ldots$ has all its elements in $[a_k, b_k]$. Can you say if this sequence converges, or has a convergent subsequence?

(c) Now suppose $\langle x_n \rangle$ is any sequence of real numbers. Show that there exists a sequence of nested intervals $[a_n, b_n]$ as in part (b) with the property that $x_n \notin [a_n, b_n]$ all every n. Then use part (b) to conclude that there exists a number $y \in \mathbb{R}$ which is not equal to x_n for any n.

(d) If you tried to use a variation on the above argument to prove that $\mathbb{Q}$ is not countable, explain at which step it would fail.

UCL Math 1101 (Analysis 1) — Fall 2011 — C. Wendl

HOMEWORK 6
Due: 21 November, 2011

Problems marked with (∗) will be graded. Students are advised to work through ALL problems in the assignments, so as to keep pace with the material.

1. (*) Compute the sum
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}.$$

2. In this problem, we denote repeating decimal expansions by writing a line over the digits that repeat, so for instance $0.\overline{3} = 0.33333333\ldots$ and $1.53\overline{678} = 1.53678678678\ldots$.

   (a) Show that $1.\overline{9} = 2$.

   (b) (*) Rewrite $1.2\overline{34}$ as a fraction.

   (c) More generally, an arbitrary repeating decimal expansion can always be written in the form
   $$\pm a_1 \ldots a_k \,.\, b_1 \ldots b_m \overline{c_1 \ldots c_n},$$
   for some integers $k, m, n \geq 0$ and $a_i, b_i, c_i \in \{0, \ldots, 9\}$. (*Caution: expressions such as* $a_1 \ldots a_k$ *in this context are not meant to be understood as products* $a_1 \cdot \ldots \cdot a_k$*, but rather as integers whose digits are* $a_1, \ldots, a_k$*.*) Note that this also includes all *terminating* decimal expansions, as the special case with $n = 0$. With this understood, show that all repeating decimal expansions represent *rational* numbers.

3. Recall that every finite sum of rational numbers is also rational. Is this also true for infinite series? That is, if $a_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and $\sum_n a_n$ converges, must its sum be rational? Prove your answer.

4. Suppose $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences with $a_n, b_n > 0$ for all n, and there is a real number $\ell > 0$ such that
$$\frac{a_n}{b_n} \to \ell$$
as $n \to \infty$. Then show:

   (a) (*) There exists a number $N > 0$ such that
   $$\frac{\ell}{2} b_n < a_n < \frac{3\ell}{2} b_n$$
   for all $n > N$.

   (b) (*) $\sum_n a_n$ converges if and only if $\sum_n b_n$ converges. This useful observation is known as the *limit comparison test.*

   Hint: use the comparison test.

   (c) The limit comparison test as stated above cannot be applied if $a_n/b_n \to 0$. Indeed, find an example where $a_n, b_n > 0$, $a_n/b_n \to 0$ and $\sum_n a_n$ converges while $\sum_n b_n$ diverges. What is the most general correct statement one could prove in the case where $a_n, b_n > 0$ and $a_n/b_n \to 0$?

5. For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges. You may use any of the results or convergence tests discussed in lecture or in Binmore, as well as the *limit comparison* test discussed in Problem 4 above.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^3 + 2}$

(b) (*) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2n+2}$

(c) (*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$

(d) (*) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)(-1)^n$

6. Another important convergence test is the *integral test*; we haven't discussed it in this course since it requires integration, a topic that will be discussed in detail in Analysis 2. For now, we shall illustrate the integral test on the example of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

for rational numbers $p > 0$. We've seen that this series converges when $p > 1$ and diverges when $p = 1$. In fact, it also diverges for all $p \leq 1$, since in this case $\frac{1}{n^p} \geq \frac{1}{n}$, so the case of $p = 1$ implies the rest via the comparison test.

To understand the convergence/divergence of these series in terms of integration, one only needs to know the following about integrals: given constants $a < b$ and a function f with $f(x) > 0$ for $x \in [a, b]$, the integral

$$\int_a^b f(x)\,dx$$

is a positive number, equal to the area of the region in the xy–plane bounded by the curves $y = f(x)$, $y = 0$, $x = a$ and $x = b$. One can also define this for $b = \infty$ as

$$\int_a^{\infty} f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

at least if the limit exists, and if this is the case then one says that the integral *converges*. Now for any rational $p > 0$, the Fundamental Theorem of Calculus (i.e. the relationship between integrals and antiderivatives) tells us

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx = \lim_{b\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^b = \lim_{b\to\infty} \frac{b^{1-p}}{1-p} - \frac{1}{1-p}$$

if $p \neq 1$, or for the special case $p = 1$,

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b x^{-1}\,dx = \lim_{b\to\infty} \ln x|_1^b = \lim_{b\to\infty} \ln b.$$

We see that the limit exists if and only if $p > 1$, hence $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if and only if $p > 1$, in perfect analogy with the convergence of the series $\sum_n \frac{1}{n^p}$. This, as it turns out, is not a coincidence.

(a) Explain why the convergence/divergence of $\int_1^{\infty} \frac{dx}{x^p}$ implies the convergence/divergence of $\sum_n \frac{1}{n^p}$. *Hint: Since we defined integration above in purely geometric terms, the best possible argument here will be geometric in nature. Draw a picture!*

(b) Under what conditions on a function $f(x)$ can you show that there is a direct relation of this sort between the convergence of $\int_1^{\infty} f(x)\,dx$ and $\sum_{n=1}^{\infty} f(n)$?

UCL Math 1101 (Analysis 1) — Fall 2011 — C. Wendl

HOMEWORK 7
Due: 28 November, 2011

Problems marked with $(*)$ will be graded. Students are advised to work through ALL problems in the assignments, so as to keep pace with the material.

1. (*) Prove that each of the following series converges. You may use any of the results or convergence tests covered in lecture on in Binmore, as well as the limit comparison test (Homework 6, Problem 4), but not the integral test (which we haven't officially covered).

   (a) $\sum_{n=1}^{\infty} \frac{n!(2n)!\sin(n^{2007})}{(3n)!}$

   (b) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$
   Hint: if you approach this the right way, you might at some point find it useful to recall that the sequence $(1+1/n)^n$ is monotone increasing and bounded above (we once showed this in lecture), hence it converges. In fact, it converges to e, but you shouldn't need that precise detail.

   (c) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$
   Hint: do you remember how to prove that $\sqrt{n+1}-\sqrt{n} \to 0$ as $n \to \infty$?

2. (*) Consider the power series

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$$

   for $x \in \mathbb{R}$. Show that there is a number $R > 0$ such that this series converges whenever $|x| < R$ and diverges whenever $|x| > R$. Compute R. (This is called the *radius of convergence* of the power series.)

3. (a) Show that if $\langle x_n \rangle$ is a sequence such that the subsequences $\langle x_{2n} \rangle$ and $\langle x_{2n-1} \rangle$ both converge to the same limit, then x_n also converges to this limit.
   Hint: we've used this fact occasionally before but never really made a point of it until now. The proof is contained in the solution to Homework 5, Problem 4(c).

   (b) (*) Consider the series

$$\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \dots$$

   This is a rearrangement of $\sum_{n=0}^{\infty} 1/2^n$, and since the latter converges absolutely, it follows from a general theorem that our series also converges, in fact to the same limit. Without using that general fact, verify directly that this series also converges. (You needn't show that its limit is the same.)
   Hint: try both the ratio test and the root test; you'll probably find that one helps and the other does not. You'll probably also need the result of part (a).

4. Use the Cauchy-Schwarz inequality to show that if the series $\sum_{n=1}^{\infty} a_n^2$ converges, then the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

   converges absolutely.

5. In the first three parts of this problem, we will fill in the details of the proof of the *alternating series test* sketched in the lecture on 17 November (the proof given in Binmore as Theorem 6.13 follows a different approach). Then we'll use the perspective gained through this argument to discuss a method for approximating π. Throughout the following, we shall refer to a series as an *admissible alternating series* if it takes the form

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

where $\langle a_n \rangle$ is any monotone decreasing sequence of nonnegative numbers with $a_n \to 0$. The statement of the alternating series test is then simply that every admissible alternating series converges.

(a) (*) Show that if $\sum_n (-1)^{n-1}a_n$ is an admissible alternating series and we denote its partial sums by

$$s_N := \sum_{n=1}^{N}(-1)^{n-1}a_n, \tag{1}$$

then the subsequence of $\langle s_N \rangle$ defined by $\langle s_{2N-1} \rangle$ is monotone decreasing and bounded below by 0.

(b) Similarly, show that the subsequence $\langle s_{2N} \rangle$ is monotone increasing and bounded above by a_1.

(c) (*) Parts (a) and (b) imply that the subsequences $\langle s_{2N-2} \rangle$ and $\langle s_{2N} \rangle$ both converge, though potentially to different limits:

$$s_{2N-1} \to L, \qquad s_{2N} \to \ell.$$

Show that in fact $\ell = L$.

Hint: We have not yet used the assumption that $a_n \to 0$; this is where you'll need it!

Once you're done with this part, Problem 3(a) implies $s_N \to \ell = L$, thus $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = \ell$ and our proof of the alternating series test is complete.

(d) Show that the sum of any admissible alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ satisfies

$$0 \le \sum_{n=1}^{\infty}(-1)^{n-1}a_n \le a_1.$$

(e) Show that if $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is an admissible alternating series summing to s and we denote its partial sums by s_N as in Equation (1) above, then for all $N \in \mathbb{N}$,

$$|s - s_N| \le a_{N+1}.$$

Hint: for any $k \in \mathbb{N}$, $a_k - a_{k+1} + a_{k+2} - \dots$ is also an admissible alternating series.

(f) You've most likely heard of Taylor series before, and you'll learn about them in more detail in Analysis II, but one specific example that's interesting to look at in the present context is

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

The equal sign in this expression should not be viewed without skepticism—the left hand side is well defined for all x, but as we've seen in lecture, the series on the right does not even converge if $|x| > 1$ (because its terms do not tend to 0), while it converges absolutely for $|x| < 1$ (by the ratio test) and conditionally for $|x| = 1$ (by the alternating series test and by comparison with $\sum_n 1/2n$). One can show that the equation is in fact valid for $-1 \le x \le 1$, and we obtain an especially interesting example by plugging in $x = 1$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots. \tag{2}$$

You can now use the result of part (e) to answer the following: how many terms in this series must one add to compute an approximation of π with error smaller than 10^{-6}?

UCL Math 1101 (Analysis 1) — Fall 2011 — C. Wendl

HOMEWORK 8
Due: 5 December, 2011

Problems marked with $(*)$ will be graded. Students are advised to work through ALL problems in the assignments, so as to keep pace with the material.

1. Evaluate each of the following limits (with proofs). You may use any of the methods covered in class, or in Chapter 8 of Binmore.

   (a) $\lim_{x\to 2} \frac{x^3+5x+7}{x^4+6x^2+8}$

   (b) $\lim_{x\to 0+} x^{1/2}$

   (c) $\lim_{x\to 3} \frac{x^2-5x+6}{x^3-27}$

   (d) (*) $\lim_{x\to 0} \frac{(1+x)^{1/2}-(1-x)^{1/2}}{x}$

2. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3-x & \text{if } x>1, \\ 1 & \text{if } x=1, \\ 2x & \text{if } x<1. \end{cases}$$

   (a) Using only the definitions, i.e., using ϵ and δ, show that

$$\lim_{x\to 1-} f(x) = 2, \qquad \lim_{x\to 1+} f(x) = 2.$$

   (b) Is the function continuous at $x = 1$?

3. (*) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3x+2 & \text{if } x\le 2, \\ x^3 & \text{if } x>2. \end{cases}$$

   Prove carefully (using ϵ and δ) that $f(x)$ is continuous at $x = 2$.

4. (*) Consider the function

$$f(x) = \begin{cases} \sin\left(\frac{2\pi}{x}\right) & \text{if } x\ne 0, \\ 0 & \text{if } x=0. \end{cases}$$

   (a) What is the largest interval containing 1 on which f is injective?

   (b) Show that f is not injective on any open interval containing 0.

   (c) Show that $\lim_{x\to 0} f(x)$ does not exist, and thus f is not continuous at 0.

   (d) Show that the function g defined by $g(x) = xf(x)$ is continuous at 0.

5. (a) A while back we proved in lecture that given any two real numbers a and b with $a < b$, there exists a rational number r with $a < r < b$. Prove that there also exists an *irrational* number ξ with $a < \xi < b$.
   Hint: you might find Homework 1 Problem 3(b) helpful.

   (b) (*) Consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x\in\mathbb{Q}, \\ 0 & \text{If } x\notin\mathbb{Q}. \end{cases}$$

   Show that for any number $a\in\mathbb{R}$, $\lim_{x\to a} f(x)$ does not exist.

UCL Math 1101 (Analysis 1) — Fall 2011 — C. Wendl

HOMEWORK 9
Due: 12 December, 2011

Problems marked with (∗) will be graded. Students are advised to work through ALL problems in the assignments, so as to keep pace with the material.

1. Consider the function defined by
$$f(x) = \frac{1}{1+x^2}.$$
Taking each of the sets listed below as the domain of f, determine in each case whether the function (i) is bounded, (ii) achieves its maximum value, (iii) achieves its minimum value.
   (a) $\mathbb{R}$
   (b) $(-1, 1)$
   (c) $(-1/3, 1]$
   (d) $(0, \infty)$
   (e) $[3, 4]$

   Decide in each case whether either of the following theorems applies:
   (i) Every continuous function on a compact interval is bounded.
   (ii) Every continuous function on a compact interval has a maximum and a minimum.

2. (*) Assume $f : [0, 1] \to \mathbb{R}$ is continuous with $f(0) = f(1)$. Show that there exists a point $c \in [0, 1/2]$ with
$$f(c) = f(c + 1/2).$$

3. (*) Show that if a function f is continuous on a compact interval $[a, b]$ and satisfies $f(x) > 0$ for all $x \in [a, b]$, then there exists a constant $c > 0$ such that
$$f(x) \ge c \quad \text{for all } x \in [a, b].$$
Informally, we say in this case that the function f is "bounded away from zero." (To help understand what this means, compare for instance the function $f(x) = x$ on the non-compact interval $(0, 1]$. It is continuous and positive on this interval, but one cannot find any constant such that $f(x) \ge c > 0$ for all $x \in (0, 1]$, since $f(x)$ gets arbitrarily close to 0 as $x \to 0$, i.e. $\lim_{x\to 0^+} f(x) = 0$.)
Give two arguments:
   (a) Prove it as a corollary of the fact that continuous functions on compact intervals are bounded. (This is easy once you see the trick!)
   (b) Prove it without using the theorem that continuous functions on compact intervals are bounded, but instead using the Bolzano-Weierstrass theorem. (See Theorem 9.11 in Binmore if you need inspiration.)

4. Just as with sequences, we can speak of functions "diverging to infinity": we say $\lim_{x\to c} f(x) = \infty$ if for every $H \in \mathbb{R}$ there exists $\delta > 0$ such that
$$0 < |x - c| < \delta \quad \Longrightarrow \quad f(x) > H,$$
and similarly, we say $\lim_{x\to c} f(x) = -\infty$ if for every $H \in \mathbb{R}$ there exists $\delta > 0$ such that
$$0 < |x - c| < \delta \quad \Longrightarrow \quad f(x) < H.$$
One should keep in mind that H may be arbitrarily large in the first case and arbitrarily small (negative!) in the second case. Here's one further variation: we say $\lim_{x\to\infty} f(x) = \infty$ if for every $H \in \mathbb{R}$ there exists $h \in \mathbb{R}$ such that
$$x > h \quad \Longrightarrow \quad f(x) > H.$$

(a) (*) Based on the pattern established by the above definitions, write down the definitions of

$$\lim_{x\to-\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = -\infty.$$

(b) (*) Prove that $\lim_{x\to c} f(x) = \infty$ if and only if for for every sequence $\langle x_n\rangle$ with x_n in the domain of f and $x_n \neq c$ for all n but $x_n \to c$,

$$f(x_n) \to \infty.$$

Hint: we've already proved the corresponding theorem when $f(x)$ has a finite limit $\lim_{x\to c} f(x) = \ell$; see Theorem 8.9 in Binmore. You can adapt the same argument for the present situation.

(c) Show that if $\lim_{x\to c} g(x) = \infty$ and $\lim_{x\to\infty} f(x) = \ell$, then

$$\lim_{x\to c} [f \circ g(x)] = \ell.$$

5. In this problem we will show that a polynomial equation of degree 3

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

always has a real solution. Here we assume $a_3 \neq 0$, so we're free to divide the equation by the leading coefficient and reduce the equation to the form

$$P(x) = x^3 + b_2x^2 + b_1x + b_0 = 0$$

for some $b_0, b_1, b_2 \in \mathbb{R}$. We proceed as follows:

(a) Show that $\lim_{x\to\infty} P(x) = \infty$ and $\lim_{x\to-\infty} P(x) = -\infty$.

(b) Using the definition of the limit, deduce from part (a) that there exist $a < 0$ and $b > 0$ such that $P(a) < 0$ and $P(b) > 0$.

(c) Apply the Intermediate Value Theorem to deduce from part (b) a solution $x \in \mathbb{R}$ to the equation $P(x) = 0$.

6. Given a subset $A \subset \mathbb{R}$ and a function $f : A \to \mathbb{R}$, we say that f is *uniformly continuous* on A if for every $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$x, y \in A \text{ with } |x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \epsilon.$$

This differs from the usual definition of continuity in two respects:

(i) For a given $\epsilon > 0$, the choice of $\delta > 0$ is not dependent on x or y but must be valid for *all* x and y in the given domain simultaneously.

(ii) We do not require A to be an interval—it may be *any* subset of $\mathbb{R}$.

Given a function $f : A \to \mathbb{R}$ that is uniformly continuous as defined above, show the following:

(a) (*) If $\langle x_n\rangle$ is a convergent sequence with $x_n \in A$ for all n, then $\langle f(x_n)\rangle$ also converges.
Hint: we are not assuming $\lim x_n \in A$. Recall that a sequence converges if and only if it is Cauchy.

(b) If $\langle x_n\rangle$ and $\langle y_n\rangle$ are two convergent sequences with $x_n, y_n \in A$ for all n and $\lim x_n = \lim y_n$, then $\lim f(x_n) = \lim f(y_n)$.

One application of uniform continuity is to make sense of the expression a^x when $a > 0$ and $x \in \mathbb{R}$ is irrational. *We know already how to define a^x when $x = p/q$ is rational: it's simply $\sqrt[q]{a^p}$. Since every irrational number can be approximated arbitrarily well by rational numbers, for any $x \in \mathbb{R}$ we can find a sequence $x_n \in \mathbb{Q}$ with $x_n \to x$ and then define*

$$a^x := \lim a^{x_n}.$$

If one can prove that $f : \mathbb{Q} \to (0, \infty) : x \mapsto a^x$ is uniformly continous, then parts (a) and (b) above imply that this limit exists and is independent of the choice of sequence converging to x. This works: one can show in fact that $f(x) = a^x$ is uniformly continuous on every bounded subset of $\mathbb{Q}$.

UCL Math 1101 (Analysis 1) — Fall 2011 — C. Wendl

HOMEWORK 10
Not Due

No problems on this sheet are marked with $(*)$, and none will be graded. Students are nonetheless advised to work through all the problems and check their solutions, because it might be fun.

1. (a) Given a subset $S \subset \mathbb{R}$ and a function $f : S \to \mathbb{R}$, we say that f is *Lipschitz continuous* on S if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \le C|x - y| \quad \text{for all } x, y \in S.$$

Show that any Lipschitz continuous function on S is also uniformly continuous on S.

(b) Show that the function $f(x) = 1/x$ is continuous but neither Lipschitz nor uniformly continuous on the interval $(0, 1]$.

(c) The function $f(x) = 1/x$ is unbounded on $(0, 1]$. Take a moment to convince yourself of this if you don't find it obvious. Then find the fatal flaw in the following argument, which is a corruption of a proof we worked through in lecture:

Dubious claim*: The function $f(x) = 1/x$ is bounded on $(0, 1]$.*

Misguided proof*: Arguing by contradiction, suppose f is unbounded on $(0, 1]$. Then there is a sequence $x_n \in (0, 1]$ such that $|f(x_n)| \to +\infty$. Since the sequence $\langle x_n \rangle$ is bounded between 0 and 1, the Bolzano-Weierstrass theorem guarantees that it has a convergent subsequence $\langle x_{j_n} \rangle$, and we shall denote its limit by*

$$x_\infty := \lim_{n\to\infty} x_{j_n}.$$

Since f is continuous on $(0, 1]$, we also have

$$f(x_{j_n}) \to f(x_\infty).$$

But this is impossible since $\langle f(x_{j_n}) \rangle$ is a subsequence of $\langle f(x_n) \rangle$ and $|f(x_n)| \to +\infty$, implying that also $|f(x_{j_n})| \to +\infty$ and thus $\langle f(x_{j_n}) \rangle$ is unbounded and cannot converge. We conclude that our initial assumption was false, and f must be bounded on $(0, 1]$.

2. A strange man with a limp stops you in the tube and asks you what you study. On hearing the answer "maths," he tells you that he's discovered a mathematical function that is sure to bring about world peace. He calls it $M(x)$ after the Russian word мир, meaning "peace". Apparently though, he doesn't know how to write down a precise formula for the function, but has managed to work out that it's continuous on $\mathbb{R}$ and satisfies the relation

$$(x^2 - 1)\,[M(x)]^4 + xM(x) = 1 \quad \text{for all } x \in \mathbb{R}. \tag{1}$$

You give the man 20p and move on, as you're certain that no such function exists. Why not?

Hint: what values must $M(x)$ have when $x = \pm 1$? Can it ever equal 0?

3. (a) Show that for every natural number n, we have $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$.

(b) Using part (a), show that for every $n \in \mathbb{N}$, the following function is continuous on $[0, \infty)$:

$$f(x) = \begin{cases} \frac{1}{x^n} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

4. We saw easily in lecture that $e^x \ge 1 + x$ for $x \ge 0$ and for $x < -1$: the first because all the terms in the series expansion for e^x are nonnegative and we kept the first two terms, the second because $e^x > 0$ while $1 + x < 0$ for $x < -1$. We shall now use the series expansion to prove that the inequality is actually satisfied for *all* x, including the interval $[-1, 0]$.

(a) As a useful lemma, show that if $\sum_n a_n$ is any absolutely convergent infinite series, then

$$\left|\sum_n a_n\right| \leq \sum_n |a_n|.$$

(b) Write $e^x = 1 + x + x^2/2 + E(x)$, where $E(x)$ denotes the sum of the remaining terms in the series. Compare $|E(x)|$ with the geometric series

$$\sum_{n=3}^{\infty} \frac{|x|^n}{2^{n-1}}$$

to show that $|E(x)| \leq |x|^3/2$ whenever $|x| \leq 1$.

(c) Show that $x^2/2 + E(x) \geq 0$ if $|x| \leq 1$, and deduce from this the inequality $e^x \geq 1 + x$.

5. (a) Use the fact that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$ to prove the inequality

$$e^x \leq \frac{1}{1-x} \text{ for all } x < 1.$$

(b) Using the two inequalities in part (a) and the sandwich theorem, show that $\lim_{x\to 0} \dfrac{e^x - 1}{x} = 1$.

(c) Show that $\lim_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$.

6. In calculus you may have learned to call a function $f(x)$ "convex" (or perhaps "concave up") when $f''(x) > 0$. Convexity is actually a much more general and important notion which can be defined without mentioning derivatives. Our goal in this problem is to show that the exponential function is convex, and also to reinterpret the arithmetic/geometric mean inequality in light of this fact.

First a definition: we say that a function $f : \mathbb{R} \to \mathbb{R}$ is *convex* if for every pair of distinct points on its graph, the line segment connecting these points is *never below* the graph of the function.

(a) Show that the geometric definition of convexity above is equivalent to the following condition: for any $a, b \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b). \tag{2}$$

(b) Prove by induction that (2) generalizes to the following: if $f : \mathbb{R} \to \mathbb{R}$ is a convex function then for any $x_1, \ldots, x_n$ and $\lambda_1, \ldots, \lambda_n$ with $\lambda_k \in [0, 1]$ and $\lambda_1 + \ldots + \lambda_n = 1$, we have

$$f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k f(x_k). \tag{3}$$

This is the classical form of a result known to analysts as *Jensen's inequality.*

(c) The inequality $e^x \geq 1 + x$ can be interpreted geometrically as the statement that the graph of $y = e^x$ never lies below the graph of the line $y = 1 + x$. Note that they do touch each other at the point $(0, 1)$, and indeed, $1 + x$ is what you've been taught in calculus to call the *tangent line* to e^x at $x = 0$. Generalize this to show that for every $a \in \mathbb{R}$, there is a line through the point (a, e^a) that is otherwise always below the graph of $y = e^x$.
Hint: the slope of the line in question will be e^a.

(d) Deduce from part (c) that e^x is a convex function.
Hint: argue by contradiction. If e^x is not convex, then you can find points $a < c < b$ such that the point (c, e^c) lies above the line connecting (a, e^a) to (b, e^b). But can you then find a line through (c, e^c) that is never above the graph of $y = e^x$?

(e) Combine the result of part (d) with inequality (3) to show that for any numbers $a_1, \ldots, a_n > 0$ and $\lambda_1, \ldots, \lambda_n \in [0, 1]$ with $\sum_{k=1}^{n} \lambda_k = 1$, we have

$$a_1^{\lambda_1} \cdot \ldots \cdot a_n^{\lambda_n} \leq \lambda_1 a_1 + \ldots + \lambda_n a_n.$$

Does this inequality look familiar in the case $\lambda_1 = \ldots = \lambda_n = 1/n$?